

EMBEDDEDNESS OF LEAST AREA MINIMAL HYPERSURFACES

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ABSTRACT. In [2], E. Calabi and J. Cao showed that a closed geodesic of least length in a two-sphere with nonnegative curvature is always simple. Using min-max theory, we prove that for some higher dimensions, this result holds without assumptions on the curvature. More precisely, in a closed $(n+1)$ -manifold with $2 \leq n \leq 6$, a least area minimal hypersurface exists and any such hypersurface is embedded.

Let (M^{n+1}, g) be a closed $(n+1)$ -manifold with $2 \leq n \leq 6$. Consider a closed connected n -manifold Γ and an immersion $\psi : \Gamma \rightarrow M$. If ψ is minimal then $\psi(\Gamma) = \Sigma$ is called a minimal hypersurface and Σ is said to be a minimal hypersurface of least area if for every minimal hypersurface $\Sigma' \subset M$, $\mathcal{H}^n(\Sigma) \leq \mathcal{H}^n(\Sigma')$. Minimal hypersurfaces which minimize the area among a family of *embedded* minimal hypersurfaces have been previously examined in some situations related to min-max theory. In [10], Marques and Neves initiated the characterisation of the area and the Morse index of surfaces produced by min-max theory in three-manifolds: among other things, they showed that in a closed oriented three-manifold with positive Ricci curvature, with Heegaard genus h and which does not contain any embedded non-orientable surface, there is an index one minimal surface Σ of genus h produced by the smooth min-max theory (see [3]) such that

$$\mathcal{H}^2(\Sigma) = \inf_{S \in \mathcal{E}_h} \mathcal{H}^2(S),$$

where \mathcal{E}_h denotes the collection of all connected embedded minimal surfaces of genus not larger than h . This result was later extended to higher dimensions by Zhou [19] in the following way. Let N^{n+1} be a closed oriented manifold with $2 \leq n \leq 6$. Let \mathcal{O} be the collection of all embedded orientable closed minimal hypersurfaces in N and \mathcal{U} be the collection of the non orientable ones. Zhou studied the minimal hypersurfaces produced by Almgren-Pitts' theory applied to the fundamental class $[N]$, under the assumption of positive Ricci curvature. It turns out in particular that their area counted with multiplicity is

$$\mathcal{A}_1(N) = \inf(\{\mathcal{H}^n(\Sigma); \Sigma \in \mathcal{O}\} \cup \{2\mathcal{H}^n(\Sigma); \Sigma \in \mathcal{U}\}).$$

Recently, Mazet and Rosenberg [13] proved that, without assumptions on the Ricci curvature, this area $\mathcal{A}_1(N)$ is realized by a hypersurface and classified all such hypersurfaces: they are either stable or produced by min-max constructions. The forementioned authors defined the quantity $\mathcal{A}_1(N)$ by separating orientable and non-orientable minimal hypersurfaces. This is due to technical reasons coming from their use of Almgren-Pitts' min-max theory. In this paper we do not need to make this distinction and we are interested in least area minimal hypersurfaces, where "least area" is taken in the geometric sense. The aim is to prove by min-max methods that in M , any least area minimal hypersurface is actually embedded. It extends a result of Calabi and Cao [2].

Theorem 1. *Let (M^{n+1}, g) be a closed $(n + 1)$ -manifold with $2 \leq n \leq 6$. Then there exists a least area minimal hypersurface $\Sigma \subset M$ and any such Σ is embedded.*

We state in Section 2 a more detailed version, which precises that, similarly to [13], the least area minimal hypersurfaces are either embedded stable hypersurfaces or given by Almgren-Pitts' min-max theory applied to the fundamental class $[M]$. Interestingly, no assumptions on the curvature are needed here whereas in dimension $n = 1$ there are counterexamples if the curvature of the sphere is not nonnegative. This is due to the fact that min-max theory produces embedded hypersurfaces for $n = 2, \dots, 6$ but only immersed geodesics for $n = 1$.

In [2], Calabi and Cao proved that on a Riemannian sphere \mathcal{S} with non-negative curvature, any non-trivial closed geodesic of the shortest length is simple. Their proof is based on the fact that if $\gamma \subset \mathcal{S}$ is a closed geodesic which is not simple, then there is a nontrivial loop $\{\sigma_t\}_{t \in [0,1]}$ in the space of one-dimensional integral current without boundary such that each σ_t has mass less than the length of γ . Then by the min-max theory of Almgren and Pitts, there exists a geodesic strictly shorter than γ . Actually they noted that, in dimension one, the min-max principle can be shown simply without Geometric Measure Theory. The construction of the "sweepout" family $\{\sigma_t\}$ in [2] relies in an essential way on the dimension of the problem: indeed for curves, there is an adhoc shortening procedure known as the Birkhoff curve shortening process. It turns out that there is already no equivalent process for surfaces. The proof of Theorem 1 will retain the main structure of the proof of Calabi and Cao, but we have to replace the Birkhoff process by a method developped by Marques and Neves in [10] when $n = 2$ and extended to $2 \leq n \leq 6$ by Zhou in [19]. The restriction on the dimension is a classical one due to the regularity results of Schoen and Simon [17]. The structure of an immersed minimal hypersurface ($n \geq 2$) is less understood than the case of geodesics, and it will add some technical issues.

This paper is organized as follows: in Section 1 are reviewed the basic notions of Almgren-Pitts' theory as well as the continuous min-max theory,

then some min-max constructions are presented and an outline of the principal proof sums up the strategy and the differences with [2]. In Section 2, the proof of Theorem 1 is given and in the Appendix, some technical results are proved.

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1. PRELIMINARIES

In this section, we present a brief overview of the essential notions of Almgren and Pitts' theory, the continuous min-max theory and some useful results. Afterwards, a heuristic description of the main steps in the proof of Theorem 1 is given as well.

1.1. Min-max theory in the setting of Almgren and Pitts

For the convenience of the reader, we give a quick review of the basic definitions from Geometric Measure Theory and some notions of the Almgren and Pitts' theory used thereafter. For a complete presentation, one can refer to the book of Pitts [16] or to Section 2 in [11].

Let M be a closed Riemannian $(n + 1)$ -manifold, assumed to be isometrically embedded in \mathbb{R}^P . We will always suppose M connected. Because we will have to consider non-orientable submanifolds, we will use flat chains modulo 2 (see [7, 4.2.26]). We will work with the space $\mathbf{I}_k(M, \mathbb{Z}_2)$ of flat chains modulo 2 in the congruence class of some k -dimensional integral currents with support contained in M , the subspace $\mathcal{Z}_k(M, \mathbb{Z}_2) \subset \mathbf{I}_k(M, \mathbb{Z}_2)$ whose elements has no boundary, and with the space $\mathcal{V}_k(M)$ of the closure, in the weak topology, of the space of k -dimensional rectifiable varifolds in \mathbb{R}^P with support in M .

As in [16], we will suppress the superscript 2 in the notation of [7]. An integral current $T \in \mathbf{I}_k(M, \mathbb{Z}_2)$ determines an integral varifold $|T|$ and a Radon measure $||T||$ ([16, Chapter 2, 2.1, (18) (e)]). If $V \in \mathcal{V}_k(M)$, denote by $||V||$ the associated Radon measure on M . Given an open set $U \subset M$, if the associated rectifiable current is an integral current in $\mathbf{I}_{n+1}(M, \mathbb{Z}_2)$, it will be written as $||U||$. To a rectifiable subset R of M corresponds an integral varifold called $|R|$. The support of a current or a measure is denoted by spt . The notation \mathbf{M} stands for the mass of an element in $\mathbf{I}_k(M, \mathbb{Z}_2)$. On $\mathbf{I}_k(M, \mathbb{Z}_2)$ there is also the flat metric $\mathcal{F}(\cdot, \cdot)$ which induces the so-called flat topology. The space $\mathcal{V}_k(M)$ is endowed with the topology of the weak convergence of varifolds. The notations $\mathcal{Z}_k(M, \underline{\underline{\nu}}, \mathbb{Z}_2)$ and $\mathbf{I}_k(M, \underline{\underline{\nu}}, \mathbb{Z}_2)$ mean that the respective spaces of currents are considered with the topology induced by $\underline{\underline{\nu}}$, where $\underline{\underline{\nu}}$ is either \mathbf{M} or \mathcal{F} .

We denote $[0, 1]$ by I . For each $j \in \mathbb{N}$, $I(1, j)$ stands for the cell complex on I whose 1-cells and 0-cells are, respectively,

$$[0, 3^{-j}], [3^{-j}, 2 \cdot 3^{-j}], \dots, [1 - 3^{-j}, 1] \text{ and } [0], [3^{-j}], \dots, [1 - 3^{-j}], [1].$$

$I(1, j)_p$ denotes the set of all p -cells in $I(1, j)$.

In the theory of Almgren and Pitts, instead of considering continuous maps from I to $\mathcal{Z}_n(M, \mathbb{Z}_2)$, one consider a sequence of mappings from $I(1, n_i)$ to $\mathcal{Z}_n(M, \mathbb{Z}_2)$, where $n_i \rightarrow \infty$ and discrete slices corresponding to adjacent vertices in $I(1, n_j)$ are closer and closer. This leads to the following notions:

Definition 2. Whenever $\phi : I(1, j)_0 \rightarrow \mathcal{Z}_n(N, \underline{\nu}, \mathbb{Z}_2)$, we define the fineness of ϕ to be

$$\mathbf{f}_{\underline{\nu}}(\phi) = \sup \left\{ \frac{\underline{\nu}(\phi(x) - \phi(y))}{\mathbf{d}(x, y)}; x, y \in I(1, j)_0, x \neq y \right\}$$

where $\mathbf{d}(x, y) = 3^j \sum_{i=1}^n |x_i - y_i|$.

For each $x \in I(1, j)_0$, define $\mathbf{n}(i, j)(x)$ to be that unique element of $I(1, j)_0$ such that

$$\mathbf{d}(x, \mathbf{n}(i, j)(x)) = \inf \{ \mathbf{d}(x, y); y \in I(1, j)_0 \}.$$

Definition 3. (1) Let $\delta > 0$. We say that ϕ_1 and ϕ_2 are homotopic in $(\mathcal{Z}_n(N, \underline{\nu}, \mathbb{Z}_2), \{0\})$ with fineness δ if and only if there exist positive integers k_1, k_2, k_3 and a map

$$\psi : I(1, k_3)_0 \times I(1, k_3)_0 \rightarrow \mathcal{Z}_n(N, \underline{\nu}, \mathbb{Z}_2)$$

such that $\mathbf{f}_{\underline{\nu}}(\psi) < \delta$ and whenever $j = 1, 2$ and $x \in I(1, k_3)_0$,

$$\phi_j : I(1, k_j) \rightarrow \mathcal{Z}_n(N, \underline{\nu}, \mathbb{Z}_2), \quad \phi_j([0]) = \phi_j([1]) = 0,$$

$$\psi([j-1], x) = \phi_j(\mathbf{n}(k_3, k_j)(x)), \quad \psi(x, [0]) = \psi(x, [1]) = 0.$$

(2) A $(1, \underline{\nu})$ -homotopy sequence of mappings into $(\mathcal{Z}_n(N, \underline{\nu}, \mathbb{Z}_2), \{0\})$ is a sequence $S = \{\phi_1, \phi_2, \dots\}$ for which there exist positive numbers $\delta_1, \delta_2, \dots$ such that ϕ_i is homotopic to ϕ_{i+1} in $(\mathcal{Z}_n(N, \underline{\nu}, \mathbb{Z}_2), \{0\})$ with fineness δ_i for each $i \in \mathbb{N}^*$, $\lim_i \delta_i = 0$ and

$$\sup \{ \mathbf{M}(\phi_i(x)); x \in \text{domain}(\phi_i), i \in \mathbb{N}^* \} < \infty.$$

(3) If $S_1 = \{\phi_i^1\}$ and $S_2 = \{\phi_i^2\}$ are $(1, \underline{\nu})$ -homotopy sequences of mappings into $(\mathcal{Z}_n(N, \underline{\nu}, \mathbb{Z}_2), \{0\})$, then S_1 is homotopic with S_2 if and only if there is a sequence $\delta_1, \delta_2, \dots$ in \mathbb{R}_+^* such that ϕ_i^1 is homotopic to ϕ_i^2 in $(\mathcal{Z}_n(N, \mathbf{M}), \{0\})$ with fineness δ_i for $i \in \mathbb{N}^*$ and $\lim_i \delta_i = 0$.

"To be homotopic with" is an equivalence relation on the set of $(1, \mathbf{M})$ -homotopy sequences of mappings into $(\mathcal{Z}_n(N, \mathbb{Z}_2), \{0\})$. An equivalence class of such sequences is an $(1, \underline{\nu})$ -homotopy classes of mappings into $(\mathcal{Z}_n(N, \underline{\nu}), \{0\})$. The space of these equivalence classes is denoted by

$$\pi_1^\#(\mathcal{Z}_n(N, \mathbb{Z}_2), \{0\}).$$

Remark 4. By [16, Theorem 4.6],

$\pi_1^\sharp(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$, $\pi_1^\sharp(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\})$ and $\pi_1(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\})$ are all naturally isomorphic.

Given $\Pi \in \pi_1^\sharp(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$, consider the function $\mathbf{L} : \Pi \rightarrow [0, \infty]$ defined such that if $S = \{\phi_i\}_{i \in \mathbb{M}} \in \Pi$ and $\phi_i : I(1, n_i) \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$, then

$$\mathbf{L}(S) = \limsup_{i \rightarrow \infty} \max\{\mathbf{M}(\phi_i(x)) : x \in I(1, n_i)\}.$$

The width of Π is then the following quantity

$$\mathbf{L}(\Pi) = \inf\{\mathbf{L}(S); S \in \Pi\}.$$

A sequence $S = \{\phi_i : I(1, n_i) \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)\}_i \in \Pi$ is said to be critical for Π if $\mathbf{L}(S) = \mathbf{L}(\Pi)$. Define the critical set $\mathbf{C}(S) \subset \mathcal{V}_n(M)$ of $S \in \Pi$ as

$$\begin{aligned} \mathbf{C}(S) &= \{V; \exists\{i_j\}_j, \exists\{x_j\}, i_j \rightarrow \infty, x_j \in I(1, n_{i_j}), \\ &V = \lim_{j \rightarrow \infty} |\phi_{i_j}(x_j)| \text{ and } ||V||(M) = \mathbf{L}(\Pi)\}. \end{aligned}$$

We recall that one can define the \mathbf{F} -metric on $\mathcal{V}_n(M)$ ([16, Chapter 2, 2.1, (19)]). The \mathbf{F} -metric induces the weak topology on any set of varifolds whose mass is bounded by a certain constant. We now give Pitts' definition of almost minimizing varifolds (see [16, Chapter 3]). Let U be an open set of \mathbb{R}^L . Denote by $\mathcal{Z}_n(M, M \setminus U, \mathbb{Z}_2)$ the set of currents $T \in \mathcal{Z}_n(M, \mathbb{Z}_2)$ such that $\partial T \subset M \setminus U$. For each pair of positive numbers ϵ and δ , we define

$$\mathbf{a}_n(U, \epsilon, \delta, \mathbb{Z}_2)$$

to be the set of all currents $T \in \mathcal{Z}_n(M, M \setminus U, \mathbb{Z}_2)$ with the following property. If $T = T_0, T_1, \dots, T_m \in \mathcal{Z}_n(N)$,

$$\begin{aligned} \text{spt}(T - T_i) &\subset U \quad \text{for } i = 1, \dots, m \\ \mathcal{F}(T_i, T_{i-1}) &\leq \delta \quad \text{for } i = 1, \dots, m \\ \mathbf{M}(T_i) &\leq \mathbf{M}(T) + \delta \quad \text{for } i = 1, \dots, m \end{aligned}$$

then $\mathbf{M}(T) - \mathbf{M}(T_m) \leq \epsilon$.

Definition 5. We say that $V \in \mathcal{V}_n(M)$ is \mathbb{Z}_2 almost minimizing in U if and only if for each $\epsilon > 0$, there exist a $\delta > 0$ and a current $T \in \mathbf{a}_n(U, \epsilon, \delta, \mathbb{Z}_2)$ such that $\mathbf{F}(V, |T|) \leq \epsilon$.

Remark 6. As we deal with flat chains modulo 2, we will use the "modulo 2" versions of results in [16], [12], [19] (see [11] for a detailed explanation). We assume that $2 \leq n \leq 6$. Note that Pitts proved [16, Theorem 7.12] for $2 \leq n \leq 5$ but because of the curvature estimates in [17, (7.2)], it still holds true for $n = 6$.

- (1) If V is a stationary n -varifold in $\mathcal{V}_k(M)$ and is \mathbb{Z}_2 almost minimizing in small annuli around each point, then $\text{spt } V$ is a smooth embedded minimal hypersurface ([11, Theorem 2.11]). This is the "modulo 2" version of Theorem 7.12 in [16].

- (2) *The interpolation results [12, Theorem 13.1], [19, Theorem 5.5] are true for \mathbb{Z}_2 (see [11, Theorem 3.9] for a statement in the "modulo 2" setting).*
- (3) *[19, Theorem 5.8] is also still true for \mathbb{Z}_2 .*

Finally, we describe the isomorphism

$$(1) \quad H_{n+1}(M, \mathbb{Z}_2) \simeq \pi_1(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\})$$

constructed in [5, Section 3]. Actually, Almgren treated the case where the coefficient group is $G = \mathbb{Z}$, but as Pitts noted in [16], all the methods extend to the case $G = \mathbb{Z}_2$. There is a number $\mu > 0$ such that if $T \in \mathbf{I}_k(M, \mathbb{Z}_2)$ has no boundary and $\mathcal{F}(T) \leq \mu$, then there is an $S \in \mathbf{I}_{k+1}(M, \mathbb{Z}_2)$ such that $\partial S = T$ and

$$\mathbf{M}(S) = \mathcal{F}(T) = \inf\{\mathbf{M}(S'); S' \in \mathbf{I}_{k+1}(M, \mathbb{Z}_2) \text{ and } \partial S' = T\}.$$

Such an S is called an \mathcal{F} -isoperimetric choice for T . Now let

$$[f] \in \pi_1(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\})$$

be a class whose one of the representative maps is

$$f : (I, \{0, 1\}) \rightarrow (\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\}).$$

The isomorphism $F : \pi_1(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\}) \rightarrow H_{n+1}(M, \mathbb{Z}_2)$ is defined as follows: take any integer m sufficiently large, there is a special chain map

$$\Phi : (I(1, m), \{[0], [1]\}) \rightarrow (\mathbf{I}_*(M, \mathbb{Z}_2), \{0\})$$

of degree m , i.e. Φ is a graded homomorphism of degree m , such that $\partial \circ \Phi = \Phi \circ \partial$, $\Phi([0]) = \Phi([1]) = 0$ and $\Phi(\alpha)$ is \mathcal{F} -isoperimetric for $\Phi(\partial\alpha)$ where $\alpha \in I(1, m)_1$. Then

$$\sum_{\alpha \in I(1, m)_1} \Phi(\alpha)$$

is a cycle in $\mathbf{I}_{n+1}(M, \mathbb{Z}_2)$. Because the homology groups of the chain complex $\mathbf{I}_*(M, \mathbb{Z}_2)$ are isomorphic with the singular homology groups of M with coefficient group \mathbb{Z}_2 , it makes sense to set

$$F([f]) = \left[\sum_{\alpha \in I(1, m)_1} \Phi(\alpha) \right] \in H_{n+1}(M, \mathbb{Z}_2).$$

1.2. Min-max theory in the continuous setting

The theory of Almgren and Pitts deals with discrete sweepouts, and this can bring some technical complications when constructing explicit sweepouts. For this reason, De Lellis and Tasnady [4] wrote a version of this theory in the continuous setting, based on ideas in [18] and [3]. In this subsection, we recall the basic notions in this setting.

Let (M^{n+1}, g) be a closed Riemannian manifold. In what follows, the topological boundary of a subset of M will be denoted by ∂ . Consider an open subset $N \subset M$ whose boundary ∂N , when non-trivial, is a rectifiable hypersurface of finite n -dimensional Hausdorff measure. Suppose also that

if ∂N is non empty, each of its connected component C separates M (i.e. $M \setminus C$ has two connected components). The notation for the m -dimensional Hausdorff measure will be \mathcal{H}^m . Take $a < b$, $k \in \mathbb{N}$.

Definition 7. A family of \mathcal{H}^n -measurable closed subsets $\{\Gamma_t\}_{t \in [a,b]^k}$ in $N \cup \partial N$ with finite \mathcal{H}^n -measure is called a generalized smooth family if

- for each t there is a finite subset $P_t \subset N$ such that $\Gamma_t \cap N$ is a smooth hypersurface in $N \setminus P_t$,
- $t \mapsto \mathcal{H}^n(\Gamma_t)$ is continuous and $t \mapsto \Gamma_t$ is continuous in the Hausdorff topology,
- $\Gamma_t \rightarrow \Gamma_{t_0}$ smoothly in any compact $U \subset\subset N \setminus P_{t_0}$ as $t \rightarrow t_0$.

If $\partial N = \emptyset$, a generalized smooth family $\{\Sigma_t\}_{t \in [a,b]}$ is called a continuous sweepout of N if there exists a family of open subsets $\{\Omega_t\}_{t \in [a,b]}$ of N such that

- (i) $(\Sigma_t \setminus \partial\Omega_t) \subset P_t$ for any $t \in (a, b]$,
- (ii) $\mathcal{H}^{n+1}(\Omega_t \setminus \Omega_s) + \mathcal{H}^{n+1}(\Omega_s \setminus \Omega_t) \rightarrow 0$, as $s \rightarrow t$,
- (iii) $\Omega_a = \emptyset$, and $\Omega_b = N$.

When $\partial N \neq \emptyset$, a continuous sweepout is required to satisfy the above conditions, except that $\partial\Omega_t$ denotes the boundary of Ω_t in N and (iii) is replaced by

- (iii) $\Omega_b = N$, $\Sigma_a = \partial N$ and $\Sigma_t \subset N$ for $t \in (a, b]$.

Still following [4], we define a notion of homotopy equivalence:

Definition 8. When $\partial N = \emptyset$, two continuous sweepouts $\{\Sigma_t^1\}_{t \in [a,b]}$ and $\{\Sigma_t^2\}_{t \in [a,b]}$ are homotopic if there is a generalized smooth family $\{\Gamma_{(s,t)}\}_{(s,t) \in [a,b]^2}$, such that $\Gamma_{(a,t)} = \Sigma_t^1$ and $\Gamma_{(b,t)} = \Sigma_t^2$. When $\partial N \neq \emptyset$, we also require the following condition: $\Gamma_{(s,t)} \subset N$ for $t \in (a, b]$ and there exists a small $\alpha > 0$ such that $\Gamma_{(s,t)} = \Gamma_{(a,t)}$ for $(s, t) \in [a, b] \times [a, a + \alpha]$.

A family Λ of continuous sweepouts is called homotopically closed if it contains the homotopy class of each of its element.

Remark 9. Little modifications are done compared with the definitions in [4] and [19] because we have to deal with non-smooth boundary ∂N .

If Λ is a homotopically closed family of continuous sweepouts, the width of N is defined as the min-max quantity

$$W(N, \partial N, \Lambda) = \inf_{\{\Sigma_t\} \in \Lambda} \max_t \mathcal{H}^n(\Sigma_t).$$

A sequence $\{\{\Sigma_t^n\}_{t \in [a,b]}\}_{n \in \mathbb{N}} \subset \Lambda$ is called a minimizing sequence if

$$\max_t \mathcal{H}^n(\Sigma_t^n) \rightarrow W(N, \partial N, \Lambda) \text{ as } n \rightarrow \infty.$$

A sequence of slices $\{\Sigma_{t_n}^n\}_{n \in \mathbb{N}}$ is called a min-max sequence if

$$\mathcal{H}^n(\Sigma_{t_n}^n) \rightarrow W(N, \partial N, \Lambda).$$

1.3. Some min-max constructions

Let (M, g) be a closed $(n + 1)$ -manifold with $2 \leq n \leq 6$. For a two-sided hypersurface Σ , the convention is that the mean curvature vector is $-H(\Sigma)\nu$, where ν is a continuous choice of unit normal vector of Σ , called outward normal. A hypersurface with boundary will be said to be generically immersed if it is the image of an immersion with normal crossings, in other words a "generic" immersion (see Definition 3.1 in [8]). Consider an open subset $N \subset M$ and suppose that if ∂N is non empty, it is a rectifiable hypersurface and each of its connected component C separates M (in particular if f is a smooth piece of ∂N , then the outwards normal is well defined).

Definition 10. *The boundary ∂N is said to be piecewise smooth mean convex if it satisfies the following property:*

- (i) *there is a generically immersed compact two-sided n -submanifold F with smooth boundary such that N is a connected component of $M \setminus F$ and*

$$\partial N \cap \partial F = \emptyset,$$

- (ii) *F has positive mean curvature at every point of ∂N with respect to any outward normal determined by N .*

More generally, a rectifiable hypersurface A is said to be piecewise smooth mean convex if there is an open set $N \subset M$ such that A is an open subset of ∂N , the first point (i) is true and the second point (ii) is satisfied for every point of A .

The first property of a piecewise smooth mean convex boundary is that it acts as a barrier for area minimizing problems.

Proposition 11. *Let (M, g) be a closed $(n + 1)$ -manifold and N an open set of M such that the boundary ∂N is non-empty and piecewise smooth mean convex. Then there are a positive number $\delta > 0$ and a vector field Z in N such that*

- (i) $\sum_{i=1}^n \langle \nabla_{e_i} Z, e_i \rangle \leq 0$ for every orthonormal family $\{e_1, \dots, e_n\}$ of vectors whose base point is in N ,
- (ii) if $p \in N$ and $d(p, \partial N) = d \leq \delta$, then

$$\langle Z(p), \frac{\partial}{\partial s} \Big|_{s=d} \exp_q(-s\nu) \rangle > 0,$$

where q is any point of ∂N such that $d(p, q) = d$ and ν is the outward normal of ∂N at q (which is well defined).

Consequently, if Σ is a hypersurface in N , then there is a diffeomorphism Ψ of N such that $\mathcal{H}^n(\Psi(\Sigma)) \leq \mathcal{H}^n(\Sigma)$ and $d(\Psi(\Sigma), \partial N) \geq \delta/2$.

Proof. This proposition is an extension of Lemma 2.2 in [10]. The idea for the piecewise smooth case is that, starting from the vector field constructed in [10], we will add some extra vector fields near the non-smooth parts of

∂N so that the final vector field is always pointing inward on ∂N . Recall that on smooth pieces of ∂N , the outward normal is well determined. By hypothesis, there is a two-sided hypersurface F such that N is a connected component of $M \setminus F$ and has positive mean curvature at every smooth point of ∂N with respect to the outward normal determined by N . By reducing F if necessary, we can suppose that the mean curvature of F is positive everywhere. Let's consider an immersed compact hypersurface $S \subset M$ with boundary endowed with a continuous choice of outward normal vector ν_S such that

$$(2) \quad N \cap S = \emptyset \text{ and } \partial N \cap \partial S = \emptyset,$$

(for instance F) and suppose that S has positive mean curvature with respect to ν_S . We can associate to S a positive real number $a(S)$ such that

$$\begin{aligned} \text{exp} : [0, 2a(S)] \times S &\rightarrow M \\ \text{exp}(r, x) &= \exp_x(-r\nu_S) \end{aligned}$$

is a local diffeomorphism, and the surface with boundary $C_r = \{r\} \times S$ has positive mean curvature. As shown in [10], when S is embedded there is a vector field on $\text{exp}([0, 2a(S)] \times S)$, called $X(S)$, such that for all $p = \text{exp}(r, x)$, $X(S)_p = \psi(r) \frac{\partial}{\partial r}$ where ψ is a function positive if $r < a(S)$ and $\psi(p) = 0$ if $r > a(S)$. Moreover for every orthonormal family $\{e_1, \dots, e_n\}$ of vectors in $\text{exp}([0, 2a(S)] \times S) \cap N$,

$$(3) \quad \sum_{i=1}^n \langle \nabla_{e_i} X(S), e_i \rangle \leq 0.$$

Since the construction is local, we can define such a vector field $X(S)$ on $\text{exp}([0, 2a(S)] \times S)$ when S is merely immersed: the domain where this vector field is defined overlaps itself when F is non embedded, so here at any given point in $\text{exp}([0, 2a(S)] \times S)$, $X(S)$ is the sum of all the local contributions. By linearity, (3) remains true. Besides we will always suppose $a(S)$ chosen small enough so that $N \cap \text{exp}([0, 2a(S)] \times \partial S) = \emptyset$. Then $X(S)$ can be extended smoothly by 0 in N and we obtain by restriction a smooth vector field of N called $Y(S)$.

Now, we want to construct a vector field in N similar to the one constructed in the proof of Theorem 2.1 in [10]. The vector field $Y(F)$ seems to be a good candidate but the flow associated to this vector field in M perhaps "leaves" N around the non-smooth parts of ∂N . Thus we will modify $Y(F)$ near the parts of ∂N where smooth pieces intersect, so that the flow stays in N . For $0 \leq k \leq n-1$, let P_k be the k -dimensional part of F , namely the set of points $p \in F$ locally lying in at least $n+1-k$ distinct hypersurfaces. In fact, P_k is the image by an immersion φ_k of a k dimensional manifold \tilde{P}_k , because F is generically immersed. Note that, since F is two-sided, for all k there exists a smooth function

$$\eta_k : \tilde{P}_k \rightarrow UM \upharpoonright_{P_k}$$

such that for all $x \in \varphi_k^{-1}(P_k \cap \partial N)$, $\eta_k(x)$ is a unit vector orthogonal to $d\varphi_k(T_x \tilde{P}_k)$ and pointing strictly inwards (i.e. $\langle \eta_k(x), \nu \rangle > 0$ for each inward normal ν of a smooth piece of ∂N touching $\varphi_k(x)$). Let $\mathbf{inj}(M)$ be the injectivity radius of M . Consider $x \in \tilde{P}_k$ ($0 \leq k \leq n-1$), we use momentarily the notation $p = \varphi_k(x)$. Denote by $D_k(x)$ the $(n+1-k)$ -disk

$$\{\exp_p(rv); r < \mathbf{inj}(M) \text{ and } v \perp d\varphi_k(T_x \tilde{P}_k)\}.$$

For $r > 0$ and $0 \leq k \leq n-1$, consider

$$S_k(r) = \{y \in D_k(x); x \in \tilde{P}_k, d(y, p) \leq r^2 \text{ and } d(y, \exp_p(r\eta_k(x))) = r\}.$$

Define also $S_n = F$.

We can choose a sequence (r_0, \dots, r_{n-1}) of small enough positive numbers so that the following properties are satisfied:

- (i) for each $0 \leq k \leq n$, $S_k = S_k(r_k)$ is a smooth immersed hypersurface with boundary verifying (2),
- (ii) for $0 \leq k \leq n-1$, S_k has positive mean curvature, the outward normal being determined by $-\eta_k$,
- (iii) the $a_k = a(S_k)$ ($0 \leq k \leq n$) are chosen so that we can find positive numbers b_k ($0 \leq k \leq n$) satisfying the following conditions. $Y(S_0)$ points inwards on ∂N and points strictly inwards on $\{y \in \partial N; d(y, P_0) < b_0\}$; for $0 \leq k \leq n-1$, $Y(S_{k+1})$ points inwards on $\{y \in \partial N; d(y, P_k) \geq b_k/2\}$ and points strictly inwards on $\{y \in \partial N; d(y, P_k) \geq b_k/2, d(y, P_{k+1}) < b_{k+1}\}$.

For positive constant $\lambda_0, \dots, \lambda_{n-1}$ chosen later, set

$$Z = Y(S_n) + \sum_{j=0}^{n-1} \lambda_j Y(S_j).$$

By linearity, this vector field automatically satisfies $\sum_{i=1}^n \langle \nabla_{e_i} Z, e_i \rangle \leq 0$ for every orthonormal family $\{e_1, \dots, e_n\}$ of vectors whose base point is in N . Then we choose $\lambda_{n-1}, \dots, \lambda_0$ in this order, in the following way: first take λ_{n-1} very large so that $Y(S_n) + \lambda_{n-1}Y(S_{n-1})$ points strictly inwards at least on $\{x \in \partial N; d(x, P_{n-2}) \geq b_{n-2}/2\}$. This is possible because of point (iii) of the previous properties. Then we choose λ_{n-2} so that $Y(S_n) + \lambda_{n-1}Y(S_{n-1}) + \lambda_{n-2}Y(S_{n-2})$ points strictly inwards at least on $\{x \in \partial N; d(x, P_{n-3}) > b_{n-3}/2\}$. We continue until λ_0 , which can be chosen large enough so that $Z = Y(S_n) + \sum_{j=0}^{n-1} \lambda_j Y(S_j)$ points strictly inwards everywhere on ∂N . In this way, there is a positive constant δ such that if $p \in N$ and $d(p, F) = d \leq \delta$, then

$$\left\langle Z(p), \frac{\partial}{\partial s} \right|_{s=d} \exp_q(-s\nu) \rangle > 0,$$

where q is any point of F such that $d(p, q) = d$ and ν is the outward normal of F at q . This proves the proposition. \square

Thanks to the previous proposition, we can check the following version of the min-max theorem.

Theorem 12. *Let (M, g) be a closed $(n+1)$ -manifold with $2 \leq n \leq 6$, and N an open set of M possibly with a non empty topological boundary. When $\partial N \neq \emptyset$, assume that ∂N is piecewise smooth mean convex. Then for any homotopically closed family Λ of sweepouts of N , with*

$$W(N, \partial N, \Lambda) > \mathcal{H}^n(\partial N)$$

when $\partial N \neq \emptyset$, there exists a min-max sequence $\{\Sigma_{t_n}^n\}$ of Λ converging in the varifold sense to an embedded minimal hypersurface Σ (possibly disconnected), contained in N . Moreover $\mathcal{H}^n(\Sigma)$, if counted with multiplicities, is equal to $W(N, \partial N, \Lambda)$.

Proof. In [19], this theorem is proved for a smooth mean convex boundary ∂N (see Theorem 2.7 in [19]). Here, we can use the previous proposition to study the piecewise smooth case. Recall that the proof in [19] uses an idea of [10], where the authors construct a vector field in N only nonvanishing in a small neighborhood of ∂N and, using the associated flow, find $a > 0$ and a minimizing sequence of sweepouts $\{\{\Sigma_t^n\}_{t \in [0,1]}\}_n$ such that

$$\mathcal{H}^n(\Sigma_t^n) \geq W(N, \partial N, \Lambda) - \delta \Rightarrow d(\Sigma_t^n, \partial N) \geq a/2,$$

where $\delta = \frac{1}{4}(W(N, \partial N, \Lambda) - \mathcal{H}^n(\partial N)) > 0$ and d is the distance function in M . Consider the vector field Z constructed in Proposition 11, then the flow Φ_t associated to Z sends N into N for all times and $\Phi_t(N) \subset \{p \in N; d(p, \partial N) \geq \delta/2\}$ for sufficiently large times. These properties of Z allow us to conclude as in [10, Theorem 2.1] and [19, Theorem 2.7]. \square

We will say that a hypersurface Σ is produced by Almgren-Pitts' theory with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2) \simeq \mathbb{Z}_2$ if the varifold $|\Sigma|$ belongs to the critical set $\mathbf{C}(S)$ of a sequence $S \in \Pi$, where $\Pi \in \pi_1^\sharp(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$ corresponds to the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$ by the isomorphism (1) and Remark 4. The following proposition will be useful.

Proposition 13. *Let Σ be an embedded connected unstable minimal hypersurface in M and suppose that there is no embedded stable minimal hypersurface S with $\mathcal{H}^n(S) < \mathcal{H}^n(\Sigma)$. Then there is a minimal hypersurface Γ produced by Almgren-Pitts' theory with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$ such that*

$$\mathcal{H}^n(\Gamma) \leq \mathcal{H}^n(\Sigma)$$

and if equality holds, then Σ itself is produced by Almgren-Pitts' theory with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$.

Proof. By intersection theory, if Σ does not separate, we can minimize the area in its non-trivial homology class in $H_n(M, \mathbb{Z}_2)$ and by the regularity theory for area minimizing flat chains modulo 2 ([14, Theorem 2.4]) we

obtain a stable non-trivial minimal hypersurface S with $\mathcal{H}^n(S) < \mathcal{H}^n(\Sigma)$. So actually Σ is two-sided. The procedure to produce Γ is then standard ([10],[19]). One constructs a continuous sweepout $\{\Sigma_t\}_{[0,1]}$ of M as in [19, Proposition 3.6] (whose proof does not use the orientability of M but only the fact that the hypersurface Σ is two-sided):

- (1) $\Sigma_{1/2} = \Sigma$,
- (2) $\mathcal{H}^n(\Sigma_t) \leq \mathcal{H}^n(\Sigma)$ with equality only if $t = 1/2$,
- (3) $\{\Sigma_t\}_{t \in [1/2-\epsilon, 1/2+\epsilon]}$ forms a foliation of a neighborhood of Σ .

$\{\Sigma_t\}$ determines by interpolation ([19, Theorem 5.5] and Remark 6) a homotopy sequence of mappings $S = \{\phi_i\}_{i \in \mathbb{N}}$ where

$$\phi_i : I(1, n_i)_0 \rightarrow \mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2) \text{ with fineness } \delta_i, \quad \lim_{i \rightarrow \infty} \delta_i = 0, \lim_{i \rightarrow \infty} n_i = \infty,$$

$$\text{and } \mathbf{L}(\Pi) \leq \mathcal{H}^n(\Sigma).$$

Let Π be the homotopy class of mappings of S . By [19, Theorem 5.8, Claim 3] and Remark 6, Π corresponds to the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$. By [16, Theorem 4.10], there is a varifold Γ which is \mathbb{Z}_2 almost minimizing in small annuli around each point and whose mass is $\mathbf{L}(\Pi)$. By [11, Theorem 2.11], Γ is actually a smooth minimal hypersurface with

$$\mathcal{H}^n(\Gamma) \leq \mathcal{H}^n(\Sigma).$$

Suppose that $\mathbf{L}(\Pi) = \mathcal{H}^n(\Gamma)$, which means that S is a critical sequence. We can consider a sequence of slices $\phi_i(\alpha^i)$, where $\alpha^i = [x_i] \in I(1, n_i)_0$, such that

$$\mathbf{M}(\phi_i(\alpha^i)) \rightarrow \mathbf{L}(\Pi).$$

Because of [19, Theorem 5.5 (1)] and Remark 6, necessarily $x_i \rightarrow 1/2$ so $\phi_i(\alpha^i)$ converge to $\Sigma_{1/2} = \Sigma$ in the flat topology by [19, Theorem 5.5 (3)]. It is known that if $T_j \in \mathcal{Z}_k(M, \mathbb{Z}_2)$ converge to $T \in \mathcal{Z}_k(M, \mathbb{Z}_2)$ in the flat topology and the sequence of varifolds $|T_j|$ converge to $V \in \mathcal{V}_k(M)$, then $||V||_k(M) = \mathbf{M}(T)$ implies that $V = |T|$ (see [16, Chapter 2, 2.1, (18), (f)]). It follows that if

$$\lim_i |\phi_i(\alpha^i)| = V \text{ and } ||V||_k(M) = \mathbf{L}(\Pi) = \mathcal{H}^n(\Sigma)$$

then $V = |\Sigma|$. Thus the only element of the critical set $\mathbf{C}(S)$ is $|\Sigma|$. This shows that Σ itself is produced by Almgren-Pitts' theory with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$. □

1.4. Outline of the proof of Theorem 1

M is a closed $(n+1)$ -manifold M with $2 \leq n \leq 6$. The main step of the proof is to show that if a least area hypersurface exists, then it is necessarily embedded (Proposition 21). This step is essentially an extension of the idea in [2] to higher dimensions. Given a minimal hypersurface Σ which is not embedded and with no stable embedded minimal hypersurface of less area, we construct an "optimal" non-trivial sweepout corresponding to Σ

in the sense that Σ is the middle slice of a mapping $A : I \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ continuous in the flat topology and other slices have area strictly less than $\mathcal{H}^n(\Sigma)$. This way of thinking Σ as the middle slice of an optimal sweepout in order to compare with other sweepouts and deduce properties about the embeddedness, the index, the area or the multiplicity of Σ appears in [2], [10], [19] and [13]. We will first show that there is a partition of the family of connected components of $M \setminus \Sigma$ into two classes \mathcal{C}_1 and \mathcal{C}_2 , such that

$$\Sigma = \partial\left(\bigcup_{c \in \mathcal{C}_i} c\right) \text{ for } i \in \{1, 2\}.$$

Roughly speaking, A is then obtained by retracting Σ to \emptyset in two different ways corresponding to \mathcal{C}_1 and \mathcal{C}_2 . The construction of this 1-parameter family of currents is based on Theorem 12. Then we use the interpolation proved by Marques and Neves in [12] for $n = 2$, later checked for higher dimensions by Zhou [19]. It enables to obtain, from a family of currents continuous in the flat topology, a homotopy sequence of mappings, to which one can apply Pitts' theory with some precise control on the obtained discrete slices. This interpolation theorem applied to A will give the wanted S . The latter belongs to a homotopy class of mappings into $(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$ called Π such that

$$0 < \mathbf{L}(\Pi) < \mathcal{H}^2(\Sigma).$$

The theory of Almgren and Pitts then produces an embedded minimal hypersurface whose area is strictly less than $\mathcal{H}^n(\Sigma)$, which completes the main step. In these arguments, we actually use flat chains modulo 2 (see Remark 6) because we work with non necessarily orientable submanifolds.

To illustrate what are the two ways of retracting Σ , let's consider the following 1-parameter family of currents $\varphi : [-1, 1] \rightarrow \mathcal{Z}_n(\mathbb{R}^{n+1})$:

$$\varphi(t) = \begin{cases} \partial\{x_1 \leq t, x_2 \leq t\} \cup \partial\{x_1 \geq -t, x_2 \geq -t\} & \text{for } t < 0 \\ \partial\{x_1 \geq t, x_2 \leq -t\} \cup \partial\{x_1 \leq -t, x_2 \geq t\} & \text{for } t \geq 0 \end{cases}$$

where x_1, x_2, \dots, x_{n+1} are the coordinate functions of \mathbb{R}^{n+1} . Let's describe what is happening in the plane generated by x_1 and x_2 . During the first half $[-1, 0[$, the boundaries of the South-West and North-East corners get closer until they meet along the "edge" $\{x = 0, y = 0\}$ and during the second half $[0, 1]$, one sees the boundaries of the North-West and South-East corners move away one from another. This family of currents will be the local model for the construction of the deformation described previously.

When one tries to extend the proof in [2] to $n \geq 2$, several technical issues arise in higher dimension. Firstly, Calabi and Cao only had to deal with curves which form an eight, whereas here Σ is not a priori so nicely immersed and we have to understand why there is still a good partition of the components of $M \setminus \Sigma$ into two classes (see Lemma 15). Secondly in order to construct the optimal sweepout, we have to use the technics in [10] and [19], but they are only developed for the smooth embedded case and

here, we have to work with components of $M \setminus \Sigma$ whose boundaries are only rectifiable (see Theorem 12 and Proposition 18).

2. PROOF OF THEOREM 1

Define the following quantities:

$$\mathfrak{A}(M) = \inf\{\mathcal{H}^n(\Sigma'); \Sigma' \subset M \text{ is a minimal hypersurface}\},$$

$$\mathfrak{A}_S(M) = \inf\{\mathcal{H}^n(\Sigma'); \Sigma' \subset M \text{ is minimal, stable and embedded}\}.$$

By definition, $\mathfrak{A}_S(M) \geq \mathfrak{A}(M)$. Contrarily to [19] and [13], we will not take care of the orientability, thus here "of least area" is to be understood in the geometric sense. We will prove the following more precise version of Theorem 1:

Theorem 14. *Let M be a closed $(n+1)$ -manifold where $2 \leq n \leq 6$. Then there exists a least area minimal hypersurface Σ_0 , i.e.*

$$\mathcal{H}^n(\Sigma_0) = \mathfrak{A}(M).$$

Such a hypersurface is either an embedded stable minimal hypersurface or a two-sided hypersurface of index one produced by the min-max theory of Almgren-Pitts with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$. In particular it is always embedded.

Let M be a closed $(n+1)$ -manifold where $2 \leq n \leq 6$. Let $\Sigma = \psi(\Gamma)$ be a minimal hypersurface, image of a connected n -manifold Γ by the immersion ψ . The latter will be supposed to be without "double cover", i.e. there is not a pair (U_1, U_2) of disjoint open sets in Γ such that $\psi(U_1) = \psi(U_2)$; since Σ is a minimal hypersurface, it is always possible to choose ψ without double cover. Denote by \mathcal{C}_Σ the set of connected components of $M \setminus \Sigma$. The complement in M of an immersed hypersurface may be quite complicated in general but it is well described if the immersion is a map with normal crossings (see [8, Definition 3.1]), that is, a "generic" immersion.

Define the closed set

$$(4) \quad E = \{x \in \Sigma; \psi^{-1}(\{x\}) \text{ has at least two distinct elements}\}$$

to be the set along which Σ self-intersects. Two minimal hypersurfaces tangentially intersecting at a point can be locally written as graphs of two functions whose difference satisfies a homogeneous elliptic equation. By the description of nodal sets for elliptic equations in [1] and [9], it is known that E is an $(n-1)$ -rectifiable set: more precisely, the set where Σ intersects itself tangentially is an $(n-2)$ -rectifiable set.

Lemma 15. *Let Σ be a non-embedded minimal hypersurface in M such that $\mathcal{H}^n(\Sigma) \leq \mathfrak{A}_S(M)$. Then there is a partition of \mathcal{C}_Σ into 2 classes \mathcal{C}_1 and \mathcal{C}_2 such that if c_1, c_2 are in \mathcal{C}_Σ , if for $p \in \Sigma \setminus E$ and $r > 0$:*

$$(B(p, r) \setminus \Sigma) \subset (c_1 \cup c_2),$$

then c_1 and c_2 are not in the same class. In particular, for $i \in \{1, 2\}$,

$$\mathcal{H}^n(\Sigma) = \sum_{c \in \mathcal{C}_i} \mathcal{H}^n(\partial c).$$

Moreover, such a partition is uniquely determined.

Proof. Let's suppose that there is no embedded stable minimal hypersurface with area strictly less than $\mathcal{H}^n(\Sigma)$. We want to construct a partition of \mathcal{C}_Σ satisfying the described property. Take a point p in an open connected component of $M \setminus \Sigma$. For all $x \in \mathcal{C}_\Sigma$, choose a smooth curve $\gamma_x : [0, 1]$ such that $\gamma_x(0) = p$, $\gamma_x(1)$ is in the interior of x and γ is generic in the sense that it only intersects Σ at a finite number of points in $\Sigma \setminus E$ and it does so transversally (this choice is possible since E is a closed set whose Hausdorff dimension is $n - 1$). Define the class of x as follows: if γ_x intersects Σ an odd number of times then declare x to be in \mathcal{C}_1 , otherwise declare it to be in \mathcal{C}_2 . It remains to show that this method gives the right partition. It suffices to show that the result of this algorithm does not depend on the choice of the paths γ_x . Let's argue by contradiction and suppose that there are two generic paths $\mu_1, \mu_2 : I \rightarrow M$ connecting p to a point q in the interior of a component $x \in \mathcal{C}_\Sigma$, but intersecting $\Sigma \setminus E$ a different number of times modulo 2. Then we can glue μ_1 and μ_2 at p and q so that we obtain a cycle intersecting generically Σ an odd number of times. It means by intersection theory that Σ represents a non-trivial class in $H_n(M, \mathbb{Z}_2)$. One can minimize the area in this homology class and eventually obtain an embedded minimal hypersurface Σ' with $\mathcal{H}^n(\Sigma') \leq \mathcal{H}^n(\Sigma)$ by [14, Theorem 2.4]. But by the assumption at the beginning of the proof, equality holds. Hence, Σ also minimizes the area in its homology class so should be embedded by [14]. This is absurd, consequently the procedure gives a good partition. Finally, it remains to check the uniqueness of such a partition (by renaming the classes if necessary). If $p_i \in M \setminus \Sigma$ and $c(p_i)$ is its connected component ($i = 1, 2$), then there is a path γ linking p_1 to p_2 and only intersecting Σ a finite number of times, transversally, and away from E . The uniqueness then comes from the fact that the class of $c(p_1)$ determines the class of each component encountered by γ , in particular $c(p_2)$. \square

If $\psi : \Gamma \rightarrow M$ is a minimal immersion of an n -dimensional manifold into (M, g) such that $\psi(\Gamma)$ is two-sided, the Jacobi operator is given by

$$L\phi = \Delta\phi + |\mathbf{A}|^2\phi + \text{Ric}_g(\nu, \nu)\phi,$$

where $\phi \in C^\infty(\Gamma)$, \mathbf{A} is the second fundamental form, and ν is a choice of outward normal of $\psi(\Gamma)$ defined on Γ . We will adopt the convention that λ is an eigenvalue of L if there exists a non-zero function ϕ such that $L\phi + \lambda\phi = 0$. Moreover, if $f \in C^\infty(\Gamma)$, define the map

$$\text{exp}_{\psi, f} : \Gamma \rightarrow M$$

$$\text{exp}_{\psi,f}(x) = \exp_{\psi(x)}(f(x)\nu(x)).$$

It is known that at each point of Γ :

$$(5) \quad \left. \frac{\partial}{\partial r} \right|_{r=0} \langle \vec{H}(\text{exp}_{\psi,rf}(\Gamma)), \nu \rangle = L(f),$$

where ν denote the natural choice of outward normal of $\text{exp}_{\psi,rf}(\Gamma)$ coming from the choice of ν on Γ .

Let $\psi : \Gamma \rightarrow M$ be an immersion of an n -dimensional manifold into (M, g) . If $\psi(\Gamma)$ is two-sided, then let $\tilde{\Gamma} = \Gamma_1 \cup \Gamma_2$ be two copies of Γ and define $\nu : \tilde{\Gamma} \rightarrow TM$ to be a continuous choice of an outward normal such that ν restricted to Γ_1 gives the opposite choice of ν restricted to Γ_2 . If $\psi(\Gamma)$ is one-sided, then let $\tilde{\Gamma}$ be a connected double cover of Γ such that there exists a continuous choice of outward normal $\nu : \tilde{\Gamma} \rightarrow TM$. Denote by $\pi : \tilde{\Gamma} \rightarrow \Gamma$ the canonical projection. A function ϕ defined on M (or Γ , or $\otimes^p T\Gamma$) lifts to a function on $\tilde{\Gamma}$ (or $\tilde{\Gamma}$, or $\otimes^p T\tilde{\Gamma}$) still denoted by ϕ . Consider the immersion $\tilde{\psi} = \psi \circ \pi$. For $f \in C^\infty(\tilde{\Gamma})$, if $\|f\|_\infty$ is small, $\text{exp}_{\tilde{\psi},f}$ is an immersion and there is a natural choice of outward normal of $\text{exp}_{\tilde{\psi},f}(\tilde{\Gamma})$ coming from ν and still called ν .

Usually if one considers a smooth two-sided unstable minimal hypersurface, one can push it using the first eigenvalue of the Jacobi operator to get a mean convex boundary which acts as a barrier for the Plateau problem for instance. The following lemma is a trick to find a hypersurface which is mean convex except perhaps in a small ball and which is close to a given minimal hypersurface. This will eventually lead to Proposition 18 which states that if Σ is non-embedded and minimal, then for any $c \in \mathcal{C}_\Sigma$ there is a piecewise smooth mean convex hypersurface approximating ∂c (see definition 10). By Theorem 11, this hypersurface acts like a barrier whether Σ is stable or not.

Lemma 16. *Consider Γ a compact n -dimensional manifold with a possibly non-empty smooth boundary and let $\psi : \Gamma \rightarrow M$ be a minimal immersion into (M, g) . Let $p \in \psi(\Gamma)$ be a point such that $\psi(\Gamma)$ is an embedded hypersurface in a neighborhood U of p . Then, using the notations previously defined, there is a metric h equal to g in $M \setminus U$ and a function*

$$f \in C^\infty(\tilde{\Gamma} \cup \partial\tilde{\Gamma})$$

positive on $\tilde{\Gamma}$ and vanishing on $\partial\tilde{\Gamma}$ such that for all $s \in (0, 1]$, $\text{exp}_{\tilde{\psi},sf}$ is an immersion and the mean curvature of $\text{exp}_{\tilde{\psi},sf}(\tilde{\Gamma})$, endowed with the natural choice of outward normal, is negative with respect to h .

Proof. Let λ be the lowest eigenvalue of the Jacobi operator L of $\tilde{\Gamma}$ (with Dirichlet condition at the boundary if $\partial\tilde{\Gamma} \neq \emptyset$) endowed with the outward normal ν . We have

$$\lambda = \inf_{\tilde{\Gamma}} \int |\nabla u|^2 - (|\mathbf{A}|^2 + \text{Ric}(\nu, \nu))u^2 d\text{vol}_g$$

where the infimum is taken among all functions $u \in H^1(\tilde{\Gamma})$ ($H_0^1(\tilde{\Gamma})$ if $\partial\tilde{\Gamma} \neq \emptyset$) of L^2 -norm one. We argue that by perturbing the metric in U if necessary, we can make $\tilde{\Gamma}$ unstable. More precisely, we will construct a metric h coinciding with g outside U such that ψ is still minimal and the lowest eigenvalue for the Jacobi operator computed with h , called λ_1 , is negative. Define the conformally changed new metric $h = \exp(2\varphi)g$, where $\varphi \in C^\infty(M)$ is will be defined later. Let \mathbf{A}_h denote the second fundamental form with respect to h . One can check the following formula:

$$(6) \quad \mathbf{A}_h(a, b) = \exp(\varphi)(\mathbf{A}(a, b) + g(a, b)d\varphi(\nu)) \quad \forall a, b \in T\tilde{\Gamma}.$$

Since $\psi(\Gamma)$ is embedded in the neighborhood U of p , it is not difficult to see that there exist a sequence of radii $\{r_i\}$ converging to zero and diffeomorphisms $\Phi_i : B_{\mathbb{R}^{n+1}}(0, 3) \rightarrow B(p, r_i) \subset M$ such that

$$(7) \quad \frac{1}{r_i^2} \Phi_i^* g \xrightarrow{i \rightarrow \infty} \frac{1}{9} g_{st},$$

$$(8) \quad \Phi_i(\mathbb{R}^n \cap B_{\mathbb{R}^{n+1}}(0, 3)) = \psi(\Gamma) \cap B(p, r_i) \text{ and } (\Phi_i^* g)_x \left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_{n+1}} \right) = 0 \quad \forall x \in \mathbb{R}^n \cap B_{\mathbb{R}^{n+1}}(0, 3), \forall k \in \{1, \dots, n\}.$$

in the C^2 topology, where \mathbb{R}^n denotes the subset $\{(x_1, \dots, x_n, 0)\} \subset \mathbb{R}^{n+1}$ and g_{st} is the standard metric on \mathbb{R}^{n+1} . Consider the function

$$\varrho : B_{\mathbb{R}^{n+1}}(0, 3) \rightarrow \mathbb{R}$$

$$\varrho(x) = \frac{2}{1 + |x|_{st}^2}$$

where $|\cdot|_{st}$ denotes the standard norm on \mathbb{R}^{n+1} . Using the previous charts, we can define a conformally changed metric $h_i = \exp(2\phi_i)g$ for each i by choosing a smooth function ϕ_i vanishing outside $B(p, r_i)$ and by imposing:

$$\phi_i = (\Phi_i^{-1})^* \log(\varrho) \text{ on } \Phi_i(B_{\mathbb{R}^{n+1}}(0, 2)),$$

$$d\phi_i(\nu) = 0 \text{ on } \psi(\Gamma) \cap B(p, r_i).$$

By (8) and (6), the previous two conditions are consistent and ψ remains a minimal immersion into M with respect to h_i for all i . Note that $\varrho^2 g_{st}$ on $B_{\mathbb{R}^{n+1}}(0, 2)$ is the metric of constant curvature one (then $B_{\mathbb{R}^{n+1}}(0, 1)$ corresponds to one hemisphere of the unit $(n+1)$ -sphere) and $\mathbb{R}^n \cap B_{\mathbb{R}^{n+1}}(0, 2)$ is an unstable minimal hypersurface with boundary for this metric. Hence, since by (7)

$$\frac{1}{r_i^2} \Phi_i^* h_i = \frac{\varrho^2}{r_i^2} \Phi_i^* g \xrightarrow{i \rightarrow \infty} \frac{\varrho^2}{9} g_{st} \text{ on } B_{\mathbb{R}^{n+1}}(0, 2),$$

if we take $\varphi = \phi_i$ for i sufficiently large, then $\tilde{\Gamma}$ contains an unstable hypersurface with boundary for $h = \exp(2\varphi)g$ so it is unstable itself for h .

Now that we have constructed h such that $\lambda_1 \neq 0$, and using (5), we can take a positive eigenfunction f for λ_1 which is in $C^\infty(\tilde{\Gamma} \cup \partial\tilde{\Gamma})$ by [15] and with sufficiently small L^∞ norm so that for all $s \in (0, 1]$, each hypersurface

$\tilde{\text{exp}}_{\tilde{\psi}, sf}(\tilde{\Gamma})$ is immersed and has negative mean curvature for h . This finishes the proof. \square

In what follows, we consider an immersed minimal hypersurface $\Sigma = \psi(\Gamma)$, where Γ is a connected closed n -manifold. Recall the following definition, introduced in [12]. If $\Phi : [a, b] \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ is continuous in the flat topology, we define

$$\mathbf{m}(\Phi, r) = \sup\{|\Phi(x)|(|B_r(p)|); x \in [a, b], p \in M\}.$$

We will say that Φ satisfies the technical condition $[\ast]$ if

$$\mathbf{m}(\Phi, r) \rightarrow 0 \text{ as } r \rightarrow 0.$$

We will need the next lemma to get rid of a certain subset of $c \in \mathcal{C}_\Sigma$ and decrease the area of ∂c inside c . The non-embeddedness of Σ is necessary.

Lemma 17. *Suppose that Σ is a non-embedded minimal hypersurface in M . Consider $c \in \mathcal{C}_\Sigma$. Then there is a map $\xi : [0, 1] \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ continuous in the flat topology such that for all $s \in [0, 1]$, $\xi(s) = \partial[G_s]$, where G_s are open sets with the following properties:*

- (i) $G_0 = c$, $\forall s$ $G_s \subset c$ and $\mathbf{M}(\xi(s)) = \mathcal{H}^n(\partial G_s)$,
- (ii) $\partial G_1 \cap c$ separates c and is piecewise smooth mean convex with respect to the normal pointing outside G ,
- (iii) $\forall s \in (0, 1]$, $\mathcal{H}^n(\partial G_s) < \mathcal{H}^n(\partial c)$,
- (iv) ξ satisfies $[\ast]$.

Proof. One knows that the set E where Σ self-intersects is a non-empty set of Hausdorff dimension $n - 1$, whereas the set where Σ intersects itself tangentially is of Hausdorff dimension $n - 2$. Thus there exists a point $q \in \partial c \cap E$ such that in a neighborhood of that point, Σ is a union of hypersurfaces intersecting two by two transversally. More precisely, there is a small radius $\delta > 0$ and a diffeomorphism $\Psi : B(q, \delta) \rightarrow \mathbb{R}^{n+1}$ such that $\Psi(q) = 0$ and $\Psi(\Sigma \cap B(q, \delta))$ can be written as union of the graphs of some functions $\omega_j : \mathbb{R}^n \rightarrow \mathbb{R}$ ($j \in \{1, \dots, L\}$), and at each point $x \in \mathbb{R}^n$, for any $j_1 \neq j_2$, either $\omega_{j_1}(x) \neq \omega_{j_2}(x)$ or $\nabla \omega_{j_1}(x) \neq \nabla \omega_{j_2}(x)$ (we identify \mathbb{R}^n with $\{(x_1, \dots, x_n, 0) \in \mathbb{R}^{n+1}\}$). Moreover, we can suppose that

$$(9) \quad \begin{aligned} \Omega &:= \{(x_1, \dots, x_n, x_{n+1}); \\ x_{n+1} &< \min\{\omega_1((x_1, \dots, x_n)), \omega_2((x_1, \dots, x_n))\}\} \subset \Psi(c \cap B(q, \delta)), \end{aligned}$$

which means that a part of ∂c around q is simply made of two half-hyperplanes meeting transversally. To check this, first consider a point $x \in \Psi(\partial c \cap E \cap B(q, \delta))$ where a minimal number of the graphs intersect. Without loss of generality, suppose that they are the graphs of $\{\omega_j\}_{1 \leq j < L_1}$ ($2 \leq L_1 \leq L$). Let's call the graph of ω_j by H_j . There is a small radius $r > 0$ such that $B(x, r)$ only intersects the graphs $\{H_j\}_{1 \leq j \leq L_1}$. Now consider the intersection

$$X = H_1 \cap H_2 \cap B(x, r)$$

which is a manifold since the two graphs meet transversally. But by minimality of L_1 ,

$$\Psi(\partial c \cap E \cap B(q, \delta)) \cap B(x, r) \subset X.$$

Consequently, there exists two integers $k_1, k_2 \in \{1, \dots, L_1\}$ such that one connected component of

$$B(x, r) \setminus (H_{k_1} \cup H_{k_2})$$

is contained in $\Psi(c \cap B(q, \delta))$. This justifies why we can suppose (9).

For simplicity, we will work in \mathbb{R}^{n+1} endowed with the metric $\tilde{g} = (\Psi^{-1})^*g$ and anything said about \mathbb{R}^{n+1} will translate into a property of $B(q, \delta) \subset M$ via the isometry Ψ . In what follows, $j = 1$ or 2 , write $\iota_j : H_j \rightarrow \mathbb{R}^{n+1}$ for the inclusion map, and ν_j for the normal vector of H_j pointing toward Ω . Because the radius δ is arbitrarily small, we can assume that $B(0, 1)$ intersects the H_j transversally. Write $\bar{\Omega} = \Omega \cup \partial\Omega$. We can apply Lemma 16 to each $B(0, 1) \cap H_j$ with a point y_j replacing p , where $y_j \in (H_j \cap B(0, 1)) \setminus \bar{\Omega}$, and U a neighborhood of y_j disjoint from $\bar{\Omega}$. In this way, we get positive functions f_j having small C^1 norm so that for $j \in \{1, 2\}$:

- $\text{exp}_{\iota_j, f_j}(B(0, 1) \cap H_j) \subset B(0, 1)$,
- in Ω , each hypersurface $\text{exp}_{\iota_j, f_j}(B(0, 1) \cap H_j)$ has negative mean curvature for the original metric, if endowed with the natural choice of outward normal given by ν_j ,
- $\text{exp}_{\iota_1, f_1}(B(0, 1) \cap H_1)$ and $\text{exp}_{\iota_2, f_2}(B(0, 1) \cap H_2)$ meet transversally.

Define for $s \in [0, 1]$

$$F_s = \{x \in \text{exp}_{\iota_j, s' f_j}(B(0, 1) \cap H_j); j \in \{1, 2\}, s' \in [0, s]\} \cap \Omega,$$

$$G_s = c \setminus \Psi^{-1}(F_s).$$

Except points (iii) and (iv), the map $\xi : s \in [0, 1] \mapsto \partial[|G_s|] \in \mathcal{Z}_n(M, \mathbb{Z}_2)$ clearly satisfies the other properties mentioned in the statement of the lemma. Point (iii) is a consequence of the first variation formula: since $H_j \cap \bar{\Omega} \cap B(0, 1)$ are minimal, only the boundary term appears and because the interior angle between these two hypersurface is less than π , we have

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{H}^n(\partial G_s) < 0,$$

so by taking f_j even smaller if necessary, (iv) is verified. Finally, point (iv) will be checked in the Appendix (see Claim 2). \square

Equipped with the previous two lemmas, we are now able to prove the following result: if Σ is non-embedded then for any $c \in \mathcal{C}_\Sigma$, the boundary ∂c can be approximated by a piecewise smooth mean convex hypersurface with less area.

Proposition 18. *Let Σ be a non-embedded minimal hypersurface in M . Consider a component $c \in \mathcal{C}_\Sigma$. Then there exists a map $\theta : [0, 1] \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ continuous in the flat topology such that for all $s \in [0, 1]$, $\theta(s) = \partial[\rho_s]$, where ρ_s are open sets with the following properties:*

- (i) $\rho_0 = c$, $\forall s$ $\rho_s \subset c$ and $\mathbf{M}(\theta(s)) = \mathcal{H}^n(\partial\rho_s)$,
- (ii) $\partial\rho_1$ is piecewise smooth mean convex,
- (iii) $\forall s \in (0, 1]$, $\mathcal{H}^n(\partial\rho_s) < \mathcal{H}^n(\partial c)$,
- (iv) there exists a homotopically closed set of continuous sweepouts of ρ_1 , called Λ ,
- (v) θ satisfies $[\ast]$.

Proof. Suppose that $\Sigma = \psi(\Gamma)$, we will use the notations previously introduced. Let ξ and $\{G_s\}_{s \in [0, 1]}$ be the map and open sets constructed in Lemma 17. Necessarily, because $\mathcal{H}^n(\partial G_1) < \mathcal{H}^n(\partial c)$ and E is of Hausdorff dimension $n - 1$, there is a point $z \in \partial c$ and a radius r_0 so that $B(z, r_0) \cap G_1 = \emptyset$ and ∂c is an embedded hypersurface in $B(z, r_0)$. Deform the metric of M inside $B(z, r_0)$ and consider the positive function f constructed in Lemma 16 with z (resp. $B(z, r_0)$) replacing p (resp. U). By [8, Chapter 3, Proposition 3.2] and Lemma 17 (ii) (iii), we can find a function \tilde{f} close in the C^∞ topology to λf where $\lambda \in (0, 1)$, so that if one defines for all $s \in [0, 1]$

$$\begin{aligned} V_s &= \{q \in \text{exp}_{\tilde{\psi}, s'}(\tilde{\Gamma}); s' \in [0, s]\}, \\ W_s &= c \setminus V_s, \\ \rho_s &= \begin{cases} G_{2s} & \text{for } s < 1/2 \\ W_{2s-1} \cap G_1 & \text{for } s \geq 1/2, \end{cases} \end{aligned}$$

then the following properties are true:

- $\partial\rho_1$ is piecewise smooth mean convex for the original metric,
- $\forall s \in [1/2, 1]$, $\mathcal{H}^n(\partial\rho_s) < \frac{1}{2}(\mathcal{H}^n(\partial G_1) + \mathcal{H}^n(\partial c)) < \mathcal{H}^n(\partial c)$.

Hence, points (ii) and (iii) follow readily. Setting $\theta(s) = \partial[\rho_s]$, point (i) is clear by construction. We will check point (iv), namely that there exists a homotopically closed set Λ of continuous sweepouts of ρ_1 , in the Appendix (see Claim 1). We will construct a continuous sweepout $\{S_t\}_{t \in I}$ of ρ_1 such that

$$S_t = \{x \in \rho_1 \cup \partial\rho_1; u(x) = t\} \text{ for any } t \in I$$

where u is a Morse function on ρ_1 continuous on $\rho_1 \cup \partial\rho_1$ with no critical points in a neighborhood of $\partial\rho_1$ and is obtained by "mollifying the distance function". Finally, the technical point (v) will be also checked in the Appendix (see Claim 2). □

Remark 19. *It should be possible to prove that, with the notations in the proof of Proposition 18, $s \mapsto \mathcal{H}^n(\partial W_s)$ is decreasing if \tilde{f} is sufficiently small. But even if it assumed to be true, we need Lemma 17 to get rid of a part of c , deform the metric g in this part and get a hypersurface being mean convex for the original metric g .*

The following lemma shows how one can retract the boundary of a $c \in \mathcal{C}_\Sigma$ to 0 and will be crucial for constructing the optimal sweepout in the proof of Proposition 21.

Lemma 20. *Let Σ be a non-embedded minimal hypersurface in M and suppose that $\mathcal{H}^n(\Sigma) \leq \mathfrak{A}_S(M)$. For any $c \in \mathcal{C}_\Sigma$, there is a map $\chi_c : I \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ continuous in the flat topology such that:*

- (i) $\chi_c(0) = \partial[|c|]$, $\chi_c(1) = 0$ and $\text{spt } \chi_c(s) \subset c$ for $s \in (0, 1]$,
- (ii) $\forall s \in (0, 1]$, $\mathbf{M}(\chi_c(s)) < \mathbf{M}(\partial[|c|]) = \mathcal{H}^n(\partial c)$,
- (iii) for j_0 large enough and for any special chain map $\Phi : I(1, j_0) \rightarrow \mathbf{I}_*(M)$ determined by $\{\chi_c(s)\}_{s \in I}$ as in [5],

$$\sum_{\alpha \in I(1, j_0)_1} \Phi(\alpha) = [|c|],$$

- (iv) χ_c satisfies $[*]$.

Proof. According to Proposition 18 (iv), we can define $W(\rho_1, \partial\rho_1, \Lambda)$. Suppose that we have

$$(10) \quad \mathcal{H}^n(\partial\rho_1) < W(\rho_1, \partial\rho_1, \Lambda).$$

By Theorem 12, there is an embedded connected minimal hypersurface $\Gamma \subset \rho_1$. We have now a manifold N whose boundary is the union of $\partial\rho_1$ and a hypersurface isometric either to Γ or to the double cover of Γ . The homology class of $\partial\rho_1$ in N is non-trivial and by [14] we can minimize the area to get an embedded stable minimal hypersurface S . Its area is not larger than $\mathcal{H}^n(\partial\rho_1)$, which is strictly smaller than $\mathcal{H}^n(\partial c)$. This is a contradiction with our assumption $\mathcal{H}^n(\Sigma) \leq \mathfrak{A}_S(M)$. So (10) is false and one can find a family $\{T_t\}_{t \in [0, 1]} \in \Lambda$ with

$$\mathcal{H}^n(T_t) < \mathcal{H}^n(\partial c) = \mathbf{M}(\partial[|c|]) \quad \forall t.$$

Define $\chi_c(s)$ to be $\theta(2s)$ if $s \in [0, 1/2]$ and the current in $\mathcal{Z}_n(M, \mathbb{Z}_2)$ determined by T_{2s-1} if $s \in [1/2, 1]$.

Let's check that the continuous map $\chi_c : I \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ satisfies the four conditions. Points (i) and (ii) in the conclusion of the theorem are clearly true by construction. Point (iii) is also true. Indeed, $\{\chi_c(s)\}_{s \in [0, 1]}$ "foliates" the open set c and one can conclude by employing the methods in [19, Theorem 5.8]. Finally the last technical condition (iv) can be proved as follows. Firstly, Lemma 18 (v) shows that

$$\sup\{||\chi_c(x)|| (B_r(p)); x \in [0, 1/2], p \in M\} \rightarrow 0 \text{ as } r \rightarrow 0.$$

Then take a $\tau \in (0, 1/2]$, Proposition 5.1 in [19] applies to $\chi_c \upharpoonright_{[1/2+\tau, 1]}$ and so

$$\sup\{||\chi_c(x)|| (B_r(p)); x \in [1/2 + \tau, 1], p \in M\} \rightarrow 0 \text{ as } r \rightarrow 0.$$

In the Appendix (see Claim 2), we will show that if τ is chosen small enough,

$$\sup\{||\chi_c(x)|| (B_r(p)); x \in [1/2, 1/2 + \tau], p \in M\} \rightarrow 0 \text{ as } r \rightarrow 0,$$

which shows that indeed $\mathbf{m}(\chi_c, r) \rightarrow 0$ as $r \rightarrow 0$. This ends the proof. \square

We are now ready to show that a least area minimal hypersurface is necessarily embedded. The regularity result [16] [Theorem 7.12] will be used in the proof. Even though it is shown by Pitts only for $2 \leq n \leq 5$, the result is still true for $n = 6$ thanks to the curvature estimates of Schoen and Simon (see [17, Section 7]). Besides each time Theorem 7.12 of [16] and Theorem 5.5 of [19] will be cited, we actually apply their "modulo 2" versions (see Remark 6).

Proposition 21. *Suppose that Σ is a non-embedded minimal hypersurface and that $\mathcal{H}^n(\Sigma) \leq \mathfrak{A}_S(M)$, then there exists a connected unstable minimal hypersurface Σ_0 such that:*

- (i) Σ_0 is embedded,
- (ii) $\mathcal{H}^n(\Sigma_0) = \mathbf{L}(\Pi) < \mathcal{H}^n(\Sigma)$, where Π is the homotopy class of mappings in $(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$ corresponding to the fundamental class in $H_{n+1}(M, \mathbb{Z}_2)$.

Proof. Let $\mathcal{C}_1, \mathcal{C}_2$ be the classes given by Lemma 15. Define $A : I \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ by

$$A(s) = \begin{cases} \sum_{c \in \mathcal{C}_1} \chi_c(1 - 2s) & \text{for } s \in [0, 1/2] \\ \sum_{c \in \mathcal{C}_2} \chi_c(2s - 1) & \text{for } s \in [1/2, 1], \end{cases}$$

where χ_c are the maps constructed in Lemma 20. It is well defined at $1/2$ because we are considering currents modulo 2. By lemma 20, this map $A : I \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ is continuous in the flat topology and

$$\sup_{x \in I} \mathbf{M}(A(x)) < \infty \text{ and } \lim_{r \rightarrow 0} \mathbf{m}(A, r) = 0.$$

Thus A determines by interpolation ([19, Theorem 5.5]) a homotopy sequence of mappings $S = \{\phi_i\}_{i \in \mathbb{N}}$ where

$$\phi_i : I(1, n_i)_0 \rightarrow \mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2) \text{ with fineness } \delta_i, \quad \lim_{i \rightarrow \infty} \delta_i = 0, \lim_{i \rightarrow \infty} n_i = \infty.$$

We want to show that S belongs to a homotopy class of mappings into $(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$, called Π , which is non-trivial. Because of [6, Theorem 13.4] (or [16, Theorem 4.6]) and [5, Theorem 8.2], this will imply $\mathbf{L}(\Pi) > 0$ so the min-max theory will produce non-trivial minimal hypersurfaces. For this purpose, recall that by [16, Theorem 4.6],

$$\pi_1^\sharp(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\}), \pi_1^\sharp(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\}) \text{ and } \pi_1(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\})$$

are all naturally isomorphic. The map A is continuous in the flat topology, so by restricting A to $I(1, n_i)$ we obtain a $(1, \mathcal{F})$ -homotopy sequence of mappings into $\pi_1^\sharp(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\})$, called \tilde{A} . Since by [19, Theorem 5.5 (3)],

$$\sup\{\mathcal{F}(\phi_i(x) - A(x)); x \in I(1, n_i)\} \leq \delta_i,$$

S determines the same $(1, \mathcal{F})$ -homotopy class of mappings into $\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\}$ as \tilde{A} , that is:

$$[S] = [\tilde{A}] \in \pi_1^\sharp(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\}).$$

$[\tilde{A}]$ is non-trivial if and only if the class $[A]$ is non-trivial in $\pi_1(\mathcal{Z}_n(M, \mathcal{F}, \mathbb{Z}_2), \{0\})$. But by Lemma 20 (iii), the method described in [5] associates to $[A]$ the homology class

$$\left[- \sum_{c \in \mathcal{C}_1} (-[c]) + \sum_{c \in \mathcal{C}_2} [c] \right] \in H_{n+1}(M, \mathbb{Z}_2) = \mathbb{Z}_2,$$

and this is equal to the non-zero fundamental class $[[M]]$, for \mathcal{C}_1 and \mathcal{C}_2 form a partition of \mathcal{C}_Σ . Consequently, the S belongs to the homotopy class of mappings Π which satisfies $\mathbf{L}(\Pi) > 0$.

As by Lemma 15 and Lemma 20 (ii), $\max_{s \in I} \mathbf{M}(A(s)) \leq \mathcal{H}^n(\Sigma)$, the interpolation theorem [19, Theorem 5.5 (1)] implies that $\mathbf{L}(\Pi) \leq \mathcal{H}^n(\Sigma)$. In the case where this inequality is strict then by [16, Theorem 4.10, Theorem 7.12], there is an embedded minimal hypersurface Σ_0 (possibly disconnected and with multiplicity) whose area is $\mathbf{L}(\Pi) > 0$. Since each connected component of Σ_0 is unstable, and by Proposition 13 has area larger than or equal to $\mathbf{L}(\Pi)$, it follows that Σ_0 is actually connected and

$$\mathcal{H}^n(\Sigma_0) = \mathbf{L}(\Pi) < \mathcal{H}^n(\Sigma).$$

The case of equality is in fact impossible. Indeed, suppose $\mathbf{L}(\Pi) = \mathcal{H}^n(\Sigma)$, then $S = \{\phi_i\}$ is a critical sequence. Consider a sequence of slices $\phi_i(\alpha^i)$, where $\alpha^i = [x_i] \in I(1, n_i)_0$, such that

$$\mathbf{M}(\phi_i(\alpha^i)) \rightarrow \mathbf{L}(\Pi).$$

Because of [19, Theorem 5.5 (1)], necessarily $x_i \rightarrow 1/2$ so $\phi_i(\alpha^i)$ converge to Σ in the flat topology by [19, Theorem 5.5 (3)]. It is known that if $T_j \in \mathcal{Z}_k(M, \mathbb{Z}_2)$ converge to $T \in \mathcal{Z}_k(M, \mathbb{Z}_2)$ in the flat topology and the sequence of varifolds $|T_j|$ converge to $V \in \mathcal{V}_k(M)$, then $||V||(M) = \mathbf{M}(T)$ implies that $V = |T|$ (see [16, Chapter 2, 2.1, (18), (f)]). It follows that if

$$\lim_i |\phi_i(\alpha^i)| = V \text{ and } ||V||(M) = \mathbf{L}(\Pi) = \mathcal{H}^n(\Sigma)$$

then $V = |\Sigma|$. Thus the only element of the critical set $\mathbf{C}(S)$ is $|\Sigma|$. But by [16, Theorem 4.3 (2), Theorem 4.10], $|\Sigma|$ should be \mathbb{Z}_2 almost minimizing in small annuli around each point, and [16, Theorem 7.12] again would imply that Σ is embedded, a contradiction. □

We can now finish the proof of the main theorem.

Proof of Theorem 1 and Theorem 14. Let's first check the existence of a least area minimal hypersurface, using arguments appearing in [13]. By the compactness result in [17], $\mathfrak{A}_S(M)$ is achieved (take a minimizing sequence and

apply the compactness theorem in balls of radius smaller than the injectivity radius of M). Thus if

$$\mathfrak{A}_S(M) = \mathfrak{A}(M),$$

the lemma is proved. Suppose on the contrary that

$$\mathfrak{A}_S(M) > \mathfrak{A}(M),$$

and take a sequence $\{\Sigma_i\}_i$ of connected minimal hypersurfaces such that $\mathcal{H}^n(\Sigma_i) < \mathfrak{A}_S(M)$ and $\lim_{i \rightarrow \infty} \mathcal{H}^n(\Sigma_i) = \mathfrak{A}(M)$. By Proposition 13 and Proposition 21, each hypersurface Σ_i has area bigger than or equal to the one of an embedded hypersurface, whose area is $\mathbf{L}(\Pi)$ where Π is the homotopy class of mappings in $(\mathcal{Z}_n(M, \mathbf{M}, \mathbb{Z}_2), \{0\})$ corresponding to the fundamental class in $H_{n+1}(M, \mathbb{Z}_2)$. Hence any minimal hypersurface produced by Almgren-Pitts' theory with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$ has area $\mathfrak{A}(M)$ in the case $\mathfrak{A}_S(M) > \mathfrak{A}(M)$.

Then Proposition 21 implies that any least area minimal hypersurface is embedded. Moreover we can use Proposition 13 to show that if Σ_0 is a least area minimal hypersurface, it is either stable or coming from Almgren-Pitts' min-max theory with the fundamental class of $H_{n+1}(M, \mathbb{Z}_2)$. Finally, if Σ_0 is not stable, we have seen in the proof of Proposition 13 that Σ_0 is two-sided and by reasoning along the lines of [10, Proposition 3.1] (see also [19], [13]), Σ_0 is indeed of index one. This finishes the proof of Theorem 14. \square

3. APPENDIX

In this section, we complete the proofs of Lemma 17, Proposition 18 and Lemma 20 by proving two claims. Before stating the claims, recall the following convention. A subset C of \mathbb{R}^{n+1} is called graph of a real function f over a domain D of a hyperplane H if for an orthonormal basis $\{e_1, \dots, e_{n+1}\}$ of \mathbb{R}^{n+1} , we have:

- $\forall i \in \{1, \dots, n\}, e_i \in H$,
- f is defined on $\tilde{D} := \{(x_1, \dots, x_n); \sum_{i=1}^n x_i e_i \in D\}$,
- $C = \{\sum_{i=1}^n x_i e_i + f(x) e_{n+1}; x = (x_1, \dots, x_n) \in \tilde{D}\}$.

Claim 1: Let N be an open subset of M such that ∂N is piecewise smooth mean convex. Then there exists a homotopically closed family Λ of sweepouts of N .

Proof of Claim 1. Recall that by definition each of the connected components of a piecewise smooth mean convex hypersurface separates M .

It suffices to find one continuous sweepout, then we will take the homotopically closed family Λ that it generates. This continuous sweepout will be determined by the level sets of a function u which will be obtained by

mollifying the distance function to ∂N , called $\mathbf{d}_{\partial N}$. Take $d_0 > 0$ smaller than the injectivity radius $\mathbf{inj}(M)$.

Consider a locally finite open cover of $U = \{p \in N : d(p, \partial N) < d_0\}$. Note that this set does not intersect ∂N . We can suppose that the cover is given by $B(p_1, a_1), B(p_2, a_2), \dots$ where

$$2a_i \leq \min\{d(p_i, \partial N), \mathbf{inj}(M)\} \quad \forall i \in \mathbb{N}^*.$$

For any i , $\exp_{p_i}^{-1} : B(p_i, 2a_i) \rightarrow \exp_{p_i}^{-1}(B(p_i, 2a_i)) \subset \mathbb{R}^{n+1}$ gives a coordinate chart. We want to modify $\mathbf{d}_{\partial N}$ via those charts. For each i , take a mollifier ϑ_i , i.e; a smooth function $\vartheta_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ with support in the ball $B_{\mathbb{R}^{n+1}}(0, 2b_i)$, positive in $B_{\mathbb{R}^{n+1}}(0, b_i)$, such that $\int_{\mathbb{R}^{n+1}} \vartheta_i(\xi) d\xi = 1$. We can assume b_i sufficiently small so that $\{x \in \mathbb{R}^{n+1}; B_{\mathbb{R}^{n+1}}(x, 2b_i) \subset \exp_{p_i}^{-1}(B(p_i, 2a_i))\}$ contains $\exp_{p_i}^{-1}(B(p_i, a_i))$. For each i , denote by $\tilde{d}^i : B(p_i, a_i) \rightarrow \mathbb{R}$ the convolution product

$$(11) \quad \tilde{d}^i(p) = \int_{\mathbb{R}^{n+1}} \vartheta_i(\exp_{p_i}^{-1}(p) - \xi) \mathbf{d}_{\partial N}(\exp_{p_i}(\xi)) d\xi.$$

It is well defined because of the choice b_i . Let ϕ_i be a partition of unity associated with $B(p_i, a_i)$. Set the function defined on $U \cup \partial N$:

$$\tilde{\mathbf{d}}(p) = \begin{cases} \sum_i \phi_i(p) \tilde{d}^i(p) & \text{for } p \in U \\ 0 & \text{for } p \in \partial N \end{cases}.$$

One can easily check that $\tilde{\mathbf{d}}$ is continuous in $N \cup \partial N$ and smooth in N . At any point p of ∂N , N is locally contained in the union of a finite number of embedded hypersurfaces intersecting at p . Hence we could have taken d_0 small enough so that for any point $p \in U \cup \partial N$, $\mathbf{d}_{\partial N}$ is locally the minimum of the distance to a finite number of embedded hypersurfaces $\Omega_1, \dots, \Omega_J$, that is for p' near p :

$$(12) \quad \mathbf{d}_{\partial N}(p') = \min_{i=1, \dots, J} d(p', \Omega_i).$$

We can reduce d_0 again if necessary, so that each function $d(\cdot, \Omega_i)$ is smooth in a neighborhood of p . Moreover because ∂N is piecewise smooth mean convex, we can find a constant $\kappa_1 > 0$ such that if $p \in \partial N$, then there is $\mathbf{v} \in T_p M$ such that $\langle \mathbf{v}, \nu \rangle_g \geq \kappa_1$, for any $\nu \in T_p M$ outward normal at p of one of the smooth pieces Ω_i . Another useful remark is that $\mathbf{d}_{\partial N}$ is differentiable almost everywhere and if it is differentiable at $p \in U$, its differential is equal to the differential of one of the $d(\cdot, \Omega_i)$ (Ω_i being as in (12)). By the usual properties of convolution with a Lipschitz function and formula (11), the differential of the function $\tilde{\mathbf{d}}$ at $p \in U \setminus \partial N$ is a local average (in a $2d(p, \partial N)$ -neighborhood of p) of the differentials of $d(\cdot, \Omega_i)$. Summing up all these facts, one obtains that

- (1) $\nabla \tilde{\mathbf{d}}$ does not vanish in U ,
- (2) $\|\nabla \tilde{\mathbf{d}}\|$ is bounded by 1 in U ,

- (3) at a point $p \in \partial N$, there is a $v(p) \in \mathbb{R}^{n+1}$ and a small radius $r(p) < \mathbf{inj}(M)$ such that for $p' \in U \cap B(p, r(p))$, $\langle d_{p'} \exp_p^{-1}(\nabla \tilde{\mathbf{d}}(p')), v(p) \rangle_{st} > \kappa_1/2$ where $\langle \cdot, \cdot \rangle_{st}$ denotes the standard scalar product in \mathbb{R}^{n+1} , and $d_{p'} \exp_p^{-1}$ is the differential of \exp_p^{-1} at p' .

Points (2) and (3) imply that there exist $K > 0$, $r_0 > 0$ verifying the following property:

$$(13) \quad \begin{aligned} & \text{for all } p \in \partial N \text{ and } p' \in B(p, r_0) \cap U, \\ & \exp_p^{-1}(B(p, r_0) \cap \tilde{\mathbf{d}}^{-1}(\tilde{\mathbf{d}}(p'))) \subset \mathbb{R}^{n+1} \\ & \text{is the graph of a } K\text{-Lipschitz function defined} \\ & \text{over a domain of a hyperplane independent of } p'. \end{aligned}$$

It remains to construct u such that it coincides with $\tilde{\mathbf{d}}$ near ∂N and is a Morse function in N . It is similar to the proof of Claim 1 in [19]. Define

$$V_s = \{p \in U \cup \partial N; \tilde{\mathbf{d}}(p) < s\}.$$

For $\epsilon > 0$ small enough, there exists a smooth function h defined on N arbitrarily close to $\tilde{\mathbf{d}}$ in $V_{2\epsilon} \setminus V_\epsilon$ for the C^1 topology, and such that $h(a) > \tilde{\mathbf{d}}(b)$ for any $a \in N \setminus V_\epsilon$ and $b \in V_{\epsilon/2}$. The function h can be assumed to be Morse because of the density of Morse functions in $C^k(N)$ for $k \geq 2$. Consider a cutoff function $\varphi : N \rightarrow \mathbb{R}$ such that $\varphi \equiv 1$ on V_ϵ , $\varphi \equiv 0$ on $N \setminus V_{2\epsilon}$ and $0 \leq \varphi \leq 1$. Define

$$u = \varphi \tilde{\mathbf{d}} + (1 - \varphi)h.$$

If h was chosen sufficiently close to $\tilde{\mathbf{d}}$ in $V_{2\epsilon} \setminus V_\epsilon$ for the C^1 topology, then point (1) in the previous list implies that ∇u does not vanish in $V_{2\epsilon}$. Reparametrizing the level sets of u , we get a family $\{\Gamma_t\}_{t \in I}$.

Note that for any smooth point p of ∂N , there is a neighborhood $B(p, r')$ of p inside which $\mathbf{d}_{\partial N}$ is smooth with non vanishing gradient and so the level sets of $\tilde{\mathbf{d}}$ come closer and closer to the level sets of $\mathbf{d}_{\partial N}$ near ∂N . Consequently, $B(p, r') \cap \tilde{\mathbf{d}}^{-1}(\tilde{\mathbf{d}}(p'))$ converge smoothly to $B(p, r') \cap \partial N$ as $p' \rightarrow p$. So adding Property (13), we readily check that $\{\Gamma_t\}_{t \in I}$ satisfies the conditions of Definition 7 and is the wanted sweepout.

□

Consider a family of closed subsets $\{C_s\}_{s \in I}$ of M . Suppose that there are a finite family of points $\{p_j\}_{j=1}^J$ and a corresponding family of radii $\{r_j\}_{j=1}^J$ such that $r_j < \mathbf{inj}(M)$, the balls $B(p_j, r_j)$ cover M and there is a positive integer K_0 such that for all $j \in \{1, \dots, J\}$ and $s \in I$,

$$\exp_{p_j}^{-1}(B(p_j, r_j) \cap C_s) \subset \mathbb{R}^{n+1}$$

is included in the union of at most K_0 graphs of K_0 -Lipschitz functions defined over domains of possibly different hyperplanes. In this case, we will say that the family $\{C_s\}$ is uniformly Lipschitz. Then we have the following

easy claim:

Claim 2: If $\Phi : [a, b] \rightarrow \mathcal{Z}_n(M, \mathbb{Z}_2)$ is continuous in the flat topology and if $\{\text{spt}(\Phi(x))\}_{x \in [a, b]}$ is a uniformly Lipschitz family of closed sets then

$$\mathbf{m}(\Phi, r) \rightarrow 0 \text{ as } r \rightarrow 0.$$

Proof of Claim 2. It is enough to verify it in each ball $B(p_j, r_j)$. But then the lemma follows from the formula for computing the area in \mathbb{R}^{n+1} of the graph of a real function defined on a domain of \mathbb{R}^n and the fact that there is a uniform constant $\kappa_2 > 0$ with $\exp_{p_j}^* g \leq \kappa_2 \cdot g_{st}$ for all j (g is the metric on M and g_{st} is the standard metric on \mathbb{R}^{n+1}). \square

The previous claim then suffices to complete the proofs of Lemma 17, Proposition 18 and Lemma 20. Indeed, in Lemma 17 and Proposition 18, the supports of ξ and θ clearly form a uniformly Lipschitz family. Secondly, for Lemma 20, we just use Property (13).

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